Under these assumptions, the invertibility principle can be expressed in the following way. If (22) is differentiated with respect to r, using (24), (25) and the assumption of constant f, the result after multiplication by σ can be written as

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right]-\frac{\zeta_{a\theta}}{\sigma}\frac{\partial\sigma}{\partial r}=\sigma\frac{\partial P}{\partial r}. \tag{26}$$

Differentiating (23) with respect to r and making use of the thermal wind equation (21), we get

$$-g\frac{\partial\sigma}{\partial r}=\frac{\partial^2 p}{\partial r\,\partial\theta}=\frac{\partial}{\partial\theta}\left[\frac{f_{\mathrm{loc}}}{R}\frac{\partial v}{\partial\theta}\right]. \tag{27}$$

(For later reference, the relation between R, σ and the static stability can be shown to be

$$R=g/(\sigma N^2\theta^2)>0, \tag{28}$$

N^2 being the static stability expressed as the square of the Brunt–Väisälä or buoyancy frequency.) Using (22) and (27), we can write (26) finally as

$$\boxed{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rv)}{\partial r}\right]+g^{-1}P\frac{\partial}{\partial\theta}\left(\frac{f_{\mathrm{loc}}}{R}\frac{\partial v}{\partial\theta}\right)=\sigma\frac{\partial P}{\partial r}} \tag{29}$$

a nonlinear equation which can be solved, for instance by relaxation methods, as described below, for the wind profile $v(r,\theta)$ given the PV distribution $P(r,\theta)$. Note that the *isentropic gradient* of P appears on the right-hand side as a prescribed forcing function.

Together with suitable boundary conditions, and the condition (17a), Eq. (29) expresses the invertibility principle in much the same way as was done in Kleinschmidt's original work. Note that if

$$f_{\mathrm{loc}}P>0, \tag{30}$$

as we shall assume, then Eq. (29) is an elliptic equation, so that the problem is well posed. As is well known, (30), together with (23), also expresses the assumption of static, inertial, and 'symmetric' baroclinic stability previously made in section 1(*d*) (e.g. Hoskins 1974, with f replaced by f_{loc}). Equation (29) is exact; its simple form is due to the assumption of circular symmetry and the use of isentropic coordinates.

Note further that if we were to make the approximations

$$f_{\mathrm{loc}}\simeq\zeta_{a\theta}\simeq f,\qquad R\simeq R_{\mathrm{ref}}(\theta),\qquad \sigma\simeq\sigma_{\mathrm{ref}}(\theta) \tag{31}$$

everywhere except when calculating the forcing function $\partial P/\partial r$ from (22), where $R_{\mathrm{ref}}(\theta)$ and $\sigma_{\mathrm{ref}}(\theta)$ are the reference-state profiles of R and σ, then (29) would simplify to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv)}{\partial r}\right)+\frac{f^2}{g\sigma_{\mathrm{ref}}}\frac{\partial}{\partial\theta}\left(R_{\mathrm{ref}}^{-1}\frac{\partial v}{\partial\theta}\right)=\sigma_{\mathrm{ref}}\frac{\partial P}{\partial r}, \tag{32}$$

which is the isentropic coordinate version of the usual quasi-geostrophic approximation to (29). The elliptic operator on the left-hand side of (32) is now linear and if, further, σ_{ref} and R_{ref} were constants, then apart from its slightly different r-dependence the operator would be a three-dimensional Laplacian, after suitably rescaling the vertical coordinate according to the Prandtl–Rossby–Burger relation

$$\Delta\theta\sim fL/(Rg\sigma)^{1/2}\qquad(\text{cf. } H\sim fL/N) \tag{33a}$$