One very noticeable result appears in each of the four tables, namely, that the radiation from the hemisphere of the sky is obtained almost exactly by an observation of the radiation at a zenith distance of 52°·30′. Zone 4. Not only does this show in the yearly means, but also in the monthly ones. The standard errors in this method of estimating the monthly mean values of H. from the readings in Zone 4, are given below:

	Table I.	Table II.	Table III.	Table IV.
Standard Error.	1·9 gr. cal.	2·0 gr. cal.	1·9 gr. cal.	2·0 gr. cal.

In the determination of these errors the fact has been allowed for, that there is, in Tables I. and III., a small difference between the yearly mean values of H. and the radiation from Zone 4. From a sample of about 100 individual observations of clear skies, being part of, or similar to those on which Table I. is based, it appears that after allowing for the difference between the yearly means, the standard error of an individual observation in determining H. from Zone 4, is 4 gr. cal. This is much smaller than would be at first expected from the magnitude of the monthly variation and the total number of the individual observations; it suggests the existence of a small seasonal variation of the difference between H. and the reading from Zone 4. The corresponding standard error of a single observation of long wave radiation from an overcast sky is about 6 gramme calories.

This result is a very fortunate one, for it renders it possible to get a good idea of the sky radiation as a whole—an important meteorological quantity—from one single observation, and will greatly facilitate the possibility of using a self-recording radiometer.

Subsequent observations in a thoroughly suitable site and locality may modify the precise value of the single zenith distance required by one or two degrees, as the instrument used at Benson was not constructed to measure altitudes to a greater nicety than one degree, and there are possibilities of a little backlash. It is noteworthy that Mr. L. F. Richardson, in his "Weather Prediction by Numerical Process," has, from theoretical considerations, given 54° as the zenith distance at which equality occurs.

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PERIODICITIES IN THE NILE FLOODS

BY

C. E. P. BROOKS, D.Sc., F.R.A.I., F.R.Met.Soc.

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Introduction.

Some years ago there came into my hands through the courtesy of Colonel Sir Henry Lyons a valuable set of Nile Flood records extending from A.D. 641 to 1451, which had been compiled from the original Coptic records and carefully corrected to the modern calendar by Mr. J. I. Craig, who kindly permitted me to make use of them. This series was admirably suited for an investigation into the vexed question of periodicity, and I determined—some time or other—to carry out such an investigation, but had not then sufficient leisure to do so by the laborious method of harmonic analysis. In 1923, in connection with another investigation, I devised a periodogram method which lessens the work required to such an extent that the preliminary analysis could be carried through as a spare-time occupation in a few months, and the results set out below are due primarily to this method.

The data are shown graphically in Fig. 1. By the deposition of the silt which it brings down, the Nile gradually raises its bed at an average rate which has been determined as 10 cm. per century. Hence the levels of the floods show a secular increase of this amount, which is indicated by the slightly sloping line in the figure. This secular variation is automatically eliminated by the difference-periodogram; in checking the results it had to be taken into account for the longer periodicities, but was of little significance for the periodicities of less than twenty-five years. After the calculations described in this paper had all been completed, a publication was received from Egypt containing another set of Nile flood levels, compiled by Prince Omar Toussoun[1] which differed in detail from those given by Mr. Craig, although the general run of the variations was the same.

Maximum readings of the Roda Gauge for each year of the two terms 1737 to 1800 and 1825 to 1872 have been published by Sir Henry Lyons,[2] and these have been utilised in some cases to check the results obtained from the longer series. The second term was brought up to 1911 from the annual reports on the Nile Flood.

The method which was employed for analysing this series of data has already been fully described under the name of the "difference-periodogram"[3]; it is briefly as follows:

(1) Divide the record into a number of equal sections and obtain the mean values *a*, *b*, *c*, *d*, *e*, etc., of each of these sections.

[1] Prince Omar Toussoun, "Mémoire sur l'histoire du Nil," *Le Caire, Mem. Inst. Egypt*, Vol. 9.

[2] Cairo, Survey Department, "The Physiography of the River Nile and its Basin," Cairo, 1906.

[3] The difference-periodogram; a method for the rapid determination of short periodicities. *London, Proc. R. Soc.*, A, **105**, 1924, p. 346.

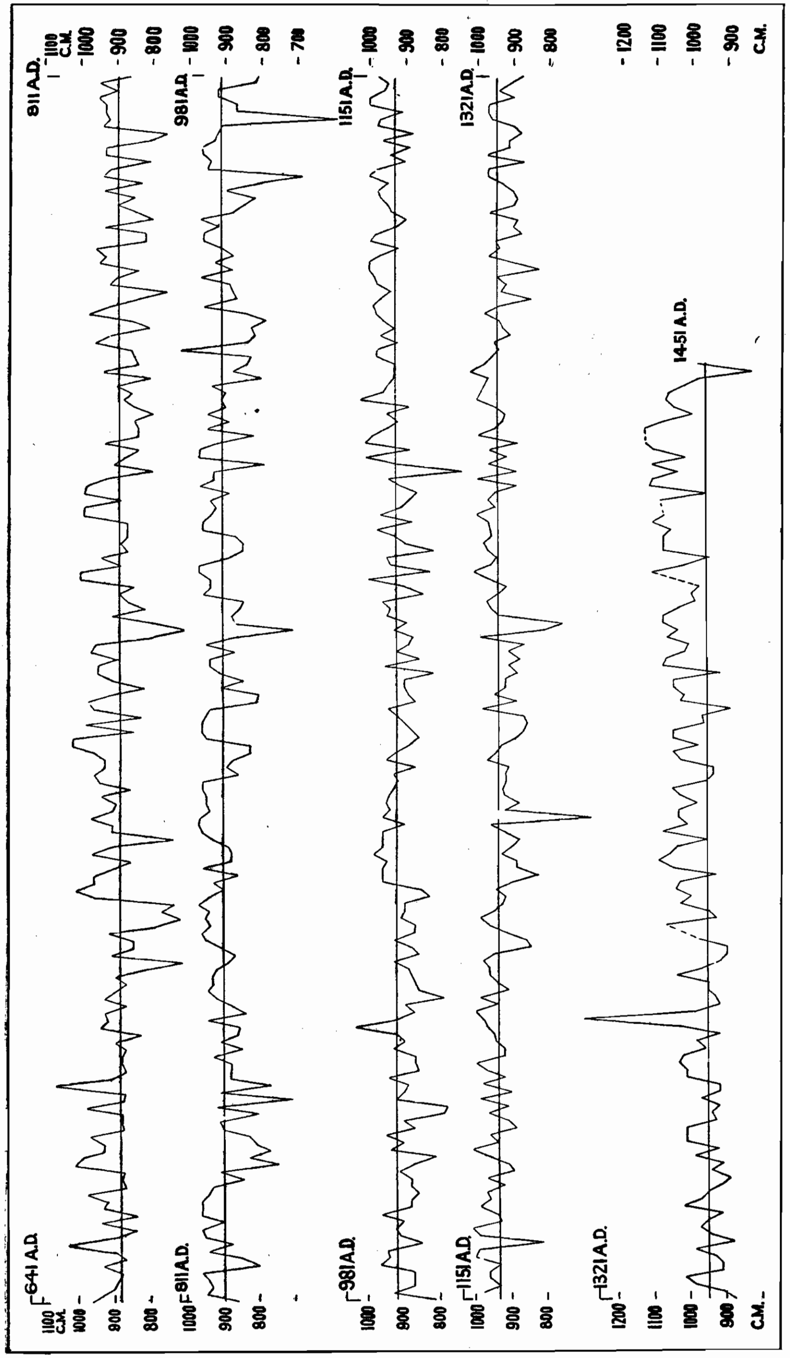

FIG. 1.—Variations in Height of Nile Flood.
The sloping line indicates the secular raising of the bed of the Nile by the deposition of silt.

(2) Form the differences $(a-b)$, $-(b-c)$, $(c-d)$, $-(d-e)$, etc.
(3) Form the means of successive differences, so that we have the "difference series":
$\frac{1}{2}\{(a-b)-(b-c)\}$, $\frac{1}{2}\{-(b-c)+(c-d)\}$, $\frac{1}{2}\{(c-d)-(d-e)\}$, etc.
(4) Plot the figures obtained in (3) with a horizontal time scale.

Suppose that the original series contains a periodicity of length P, and that the sections into which the series has been divided (see (1) above) are of length U. Then the periodicity P will be represented in the "difference series" by a new periodicity or "cycle" C, such that

$$P=\frac{2\,C\,U}{C\pm 2U}.$$

Now it can be shown that unless the length of P is between about 1·3 and 4 times the length of U, the amplitude of C will be small, so that the "difference series" gives a curve from which all periodicities except those between $1{\cdot}3U$ and $4U$ have been, practically speaking, eliminated. Any secular variation is completely eliminated. Moreover, it is obvious that the "difference series" can itself be regarded as a new series, which can be analysed in turn by the difference periodogram or by any other method such as harmonic analysis.

Owing to the alternative + or − signs of the denominator each value of C leads to two alternative values of P, only one of which is correct. If we next divide the series into sections of a slightly different length U', we obtain another pair of values of P, one of which should be identical with one of the first pair; this is the true value. Thus by taking a number of different values of U ranging from the actual time unit in which the original series is expressed (in this case one year), upwards, we can determine all periodicities with a length of 1·33 years or more. The practicable upper limit for U is one-twentieth of the whole series of observations, giving as the upper limit of the periodicities determined in this way about a hundred years.

As an example I have selected the two "difference series" obtained with $U=20$ and $U=25$. The working is given for the first of these only (Table I.).

TABLE I.—CONSTRUCTION OF DIFFERENCE-PERIODOGRAM, $U=20$.

(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
a 914	+ 1		889	− 43	− 30	932	+ 27	+ 72	939	− 17	+ 11
b 913	− 44	− 43	932	− 28	− 71	905	+ 10	+ 37	956	− 29	− 46
c 869	− 51	− 95	904	+ 6	− 22	915	− 40	− 30	927	0	− 29
d 920	− 30	− 81	898	+ 11	+ 17	955	+ 8	− 32	927	+ 16	+ 16
e 890	− 28	− 58	909	+ 18	+ 29	963	+ 10	+ 18	943	− 46	− 30
f 918	− 47	− 75	891	− 18	0	953	− 7	+ 3	989	− 19	− 65
g 871	− 8	− 55	873	− 22	− 40	946	− 6	− 13	970	− 51	− 70
h 879	+ 25	+ 17	895	− 5	− 27	952	− 50	− 56	1021	− 2	− 53
i 904	+ 28	+ 53	890	+ 3	− 2	902	− 12	− 62	1019	− 48	− 50
j 876	+ 13	+ 41	887	+ 45	+ 48	914	+ 28	+ 16	1067		

(1) = consecutive means over 20 years.
(2) = $(a-b)$, $-(b-c)$, $(c-d)$, $-(d-e)$, etc.
(3) = $\{(a-b)-(b-c)\}$, $\{-(b-c)+(c-d)\}$, $\{(c-d)-(d-e)\}$, etc.
= twice the "Difference series."

The two difference curves are shown in Fig. 2; in the difference curve for $U = 20$ two different lengths of the cycle C are shown, one with a length of $5U$, corresponding with a periodicity of 28·6 or 66·6 years, and the other with a length of the order of $35U$, which cannot be determined accurately, corresponding with a periodicity of about 39 or 42 years. The difference curve for $U = 25$ shows one main cycle with a length of $8U$, corresponding with a periodicity of either 40 or 67 years. The final result of the work is to determine for each value of U the best developed periodicities between $1{\cdot}33U$ and $4U$, to the number of one, two, or in some cases three. By taking the values of U sufficiently near together, it is ensured that all real periodicities of sufficient amplitude and stability are found.

2. The Periods found.

The values of the cycle C and the corresponding pairs of alternative periodicities obtained from the difference-periodogram with different values of U are set out in Table II., the alternative values of each pair being placed one above the other. The figures in heavy type show cycles of outstanding magnitude or regularity; those in brackets refer to rather small or irregular cycles. In the case of $U = 25$, the difference curve apparently results from the interference of two cycles of nearly equal length, suggesting that there are two periodicities, one near 40 years and the other near 67 years.

TABLE II.—Periodicities suggested by the Difference-Periodogram.

U Years.	Cycle and Periodicities.					
	C	P	C	P	C	P
1	$6{\cdot}7U$	1·539 2·854	$25U$	1·86 2·175	$40U$	1·90 2·11
2	$5U$	2·857 6·67	$25U$	3·69 4·36		
3	$12U$	5·14 7·2	**$16U$**	**5·33** **6·86**		
4	$5{\cdot}8U$	6·1 12·2	**$12U$**	**6·85** **9·6**	$68U$	7·8 8·3
5	$6{\cdot}33U$	7·6 14·7	$10U$	8·3 12·5	$19U$	9·1 11·2
6	$10{\cdot}2U$	10·0 14·9	$27{\cdot}5U$	11·2 12·9	$(55U)$	(11·58) (12·45)
8	$4{\cdot}7U$	11·2 28	$25U$	14·8 17·4	$40U$	15·2 16·8
10	$8{\cdot}2U$	16·0 26·7	$(20U)$	(18·2) (22·2)		
12	$6U$	18 36	$42U$	22·9 25·2		
15	$8{\cdot}3U$	24·2 39·5	$17{\cdot}3U$	26·8 34·0		
20	**$5U$**	**28·6** **66·6**	$(35U)$	(39) (42)		
25	**$8U$**	**40** **67**				
30	$9{\cdot}5U$	49 76	$17U$	54 68		
40	**$9U$**	**66** **103**	$35U$	76 85		

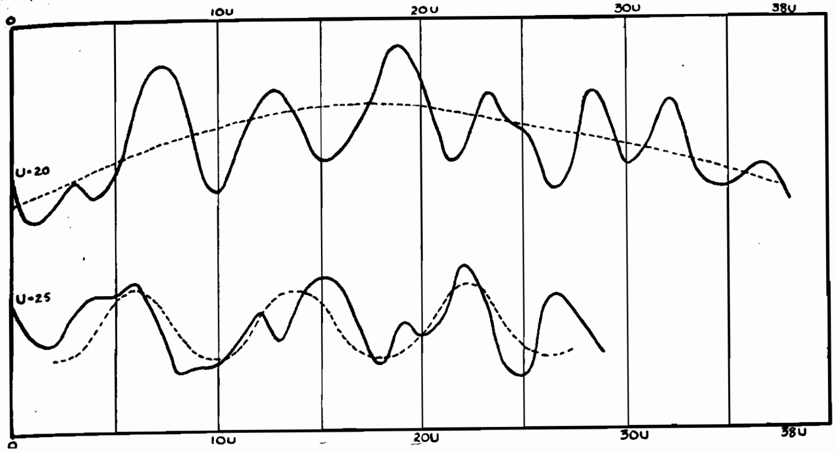

Fig. 2.—Difference-Periodogram, U = 20, U = 25.

The following periodicities were picked out, either because they occur more than once, or because they fall near periodicities well known from other series.

	Period finally determined.
1. Approximately 2 years.	(?1·91, 2·177) years.
2. 2·854, 2·857 years.	2·860 years.
3. 3·69 years.	3·66 ,,
4. 5·33 years.	5·52 ,,
5. 6·86, 6·85 years.	6·83 ,,
6. 7·17, 7·2, 7·6, mean 7·32 years.	7·33 ,,
7. 8·3, 8·3 years.	8·33 ,,
8. 11·2, 11·2, 11·2, mean 11·2 years.	10·86–11·36 years.
9. 12·6, 12·45 years.	12·50 years.
10. 14·7, 14·9, 14·8, mean 14·8 years.	14·87 ,,
11. 16·8, 16·0 years.	16·68 ,,
12. 18·2, 18·0 years.	18·32 ,,
13. 22·2, 22·9 years.	21·81–22·43 years.
14. 25·2, 24·3 years.	24·43 years.
15. 34 years.	33·49 ,,
16. 39·5, 39, 40, mean 39·5 years.	39·85 ,,
17. 66·6, 67, 66, 68, mean 66·9 years.	64·6–67·4 years.
18. 76, 76 years.	76·8 years.

The values in the last column were obtained by harmonic analysis. The whole set of data from A.D. 641 to 1451 was divided into a number of equal lengths of about 100 years, these lengths being integral multiples of some convenient period near the periodicity under investigation. For example, when the periodicity of 5·3–5·4 years was being examined, the data were divided into lengths of 88 years, or 16 times 5·5 years. The sixteenth harmonic was then calculated for each of these lengths of 88 years, and the phase angles relative to A.D. 641 were plotted. It was found that the phase decreased at an average rate of about 20° in each period of 88 years, indicating that the period of 5·5 years was too short by about $\frac{20 \times (5{\cdot}5)^2}{360 \times 88} = 0{\cdot}02$ years, *i.e.* that the real value of the periodicity is 5·52 years.

3. PERIODICITIES FOUND WITH $U=1$.

The difference curves of the first order with $U=1$ suggested values of C equal to approximately 6·5 years, 25 years and 40 years. A closer approximation was obtained by drawing difference curves of the second order (*i.e.* treating the "difference-series" W of the first order as a series of original observations), with $V=3W$, $4W$, $10W$, $15W$ and $20W$. The first two gave a value of $C' = 6{\cdot}67$; $10W$ and $15W$ gave $C' = 24{\cdot}9$ and $15W$ gave $C' = 39$. The series with $V=20W$ indicated a value of C' very close to 40, but was not long enough to determine it exactly.

In order to obtain the length, phase and amplitude of these difference cycles accurately, the values of W were written out in sets of 25 and 40 and analysed harmonically, in the form $C = a \sin (t+\phi')$, with the following results:

			a (cm.)	ϕ'	ϕ (corrected to 641).
(a) $C=6{\cdot}67$.	A.D.	642– 761	31	80	217
		762– 881	31	101	258
		882–1001	23	359	176
		1002–1121	16	61	213
		1122–1241	6	311	223
		1242–1361	29	352	235
		1362–1451	10	341	233

Here ϕ' is the phase of the derived cycle C, ϕ is the corresponding phase of the original period P.

The best-fitting straight line gives an average decrease of ϕ' by 20° in 120 years, or 18 cycles. Hence the true value of $C=6{\cdot}653$ years; the amplitude of the C cycle is 19 cm. and its phase at A.D. 641 is 227°.

			a (cm.)	ϕ'	ϕ (corrected to 641).
(b) $C=25$.	A.D.	642– 841	25	353	147
		842–1041	20	153	27
		1042–1241	15	108	62
		1242–1441	49	85	109

The best-fitting straight line shows an average decrease of 80° in 200 years or 8 cycles. Hence the true value of C is 24·3 years; the amplitude is 21 cm. and the phase at A.D. 641 is 99°.

			a (cm.)	ϕ'	ϕ (corrected to 641).
(c) $C=40$.	A.D.	642– 921	30	178	20°
		922–1201	40	197	– 24°
		1202–1451	19	317	33°

The best-fitting straight line shows an average increase of 63° in 280 years or 7 cycles. Hence the true value of $C=41{\cdot}0$ years, the amplitude is 27·4 cm. and the phase at A.D. 641 = 3°.

These values give the three pairs of alternative periodicities shown for $U=1$ in Table II. It was necessary to determine these particulars as accurately as possible from the difference-values (W) since we have only annual observations at our disposal, and consequently the direct determination of periodicities round about two years would have been very difficult. With values of U equal to 2 or more, it is not necessary to determine the length of C so exactly, since the exact length, phase and amplitude of the suspected periodicities can be determined directly from the original observations.

It will be noticed in the list of periodicities on p. 13 that those of about 11, 22 and 66 years are given a range of values instead of a single definite value. This is because these periods, and especially the two latter, show a well-marked cyclic variation in length. This was first noticed on the

difference-periodogram with $U=20$ (Fig. 2), which shows a beautiful series of waves with lengths respectively of

$$4U,\ 5\tfrac{1}{2}U,\ 6U,\ 5U,\ 5U,\ 4U,\ 5U.$$

This suggests a fluctuation in the length of the periodicity having a term of about $25U$ or some 500 years. This fluctuation was afterwards investigated more closely, and the results are discussed in section 8.

4. CLOSER DETERMINATIONS.

(*a*) *Periodicities of approximately two years.*

The periodicities of 1·85 or 2·177 years and 1·91 or 2·10 years are too near to 2 years to be shown on the difference-periodogram with $U=2$. A periodicity of 2·177 years gives a value of C of only $2{\cdot}39\,U_2$, and one of 2·11 years a value of C of $2{\cdot}22\,U_2$. When the difference-curve for U_2 is closely scrutinised and all the minor maxima and inflexions are counted, there are 103 cycles distributed as follows:

C	2	3	4	5	6	7
Number	7	41	31	13	8	3

The main periodicity in the difference-curve brought out by further analysis is $5{\cdot}0\,U$, giving a periodicity of 2·857 years, and it is evident that a shorter cycle of between 2 and $3U$ is superposed on this. Hence one at least of the two periodicities near two years has the higher of the two alternative values, that is, we have a periodicity of either 2·10 or 2·177 years, and possibly of both these lengths. These periodicities reappear in the intervals A.D. 1737–1800, 1825–1911, with average amplitudes of about 8 cm.

(*b*) *Periodicity of 2·860 years.*

This is an important periodicity, which maintains a considerable degree of constancy throughout the whole period. The amplitude and phase (reduced to A.D. 641) are as follows:

		a (cm.)	ϕ
A.D.	642– 761	12	217
	762– 881	12	258
	882–1001	9	176
	1002–1121	6	213
	1122–1241	2	223
	1242–1361	11	235
	1362–1451	4	233
	642–1451	7·4	227

In the intervals A.D. 1737 to 1800, 1825 to 1911, this periodicity reappears very markedly; its amplitude is 12 cm. from A.D. 1737 to 1800, and as much as 31 cm. from A.D. 1825 to 1911.

(*c*) *Periodicity of 3·66 years.*

This is rather variable in phase and amplitude:

		a (cm.)	ϕ
A.D.	641– 728	12	14 (374)
	729– 816	19	208
	817– 890	4	29
	905– 992	12	232
	993–1080	3	225
	1081–1168	10	95 (455)
	1169–1256	3	334
	1257–1344	4	201
	1345–1432	13	289

The mean amplitude (regarded as a vector) is only 3·1 cm., the phase at A.D. 641 is 228°. There is a suggestion of a cyclic variation in the length with a term of about 500 years, the length reaching a maximum of 3·8 years about A.D. 775 and a minimum of 3·6 years about A.D. 1025. Indications of this variation of length reappear in quite a number of the periodicities found in the course of this paper, sometimes very markedly, as in that of 22 years. The evidence for its reality is discussed in section 8; if it is accepted it means that the true expression of the periodicity is not simply $y = a \sin\left(\frac{2\pi t}{P} + \phi\right)$ but a more complex form.

If we suppose that the variation of ϕ follows a sine curve we may write:

$$y = \grave{a} \sin\left[\frac{2\pi t}{P} + \phi + b \sin\left(\frac{2\pi t}{500} + \theta\right)\right]$$

This expression may be employed to correct the amplitude of the periodicity. When this is done we find that the corrected amplitude $\grave{a}$ becomes 7 cm.

(*d*) *Periodicity of 5·52 years.*

This is half the sunspot periodicity, but is irregular in its length and amplitude.

		a (cm.)	ϕ
A.D	641– 728	13	25
	729– 816	8	6
	817– 904	2	25
	905– 992	5	137
	993–1080	3	162
	1081–1168	6	315
	1169–1256	11	75 (435)
	1257–1344	5	147
	1345–1432	30	0

The mean amplitude is 5·8 cm. and the phase at A.D. 641 is 23°. There is a fairly well marked fluctuation in the phase, with a duration of rather over 500 years.

(*e*) *Periodicity of 6·83 years.*

This is irregular both in length and amplitude.

		a (cm.)	ϕ
A.D.	641– 736	13	61
	737– 832	12	120
	833– 928	14	35
	929–1024	24	173
	1025–1120	3	332
	1121–1216	12	170
	1217–1312	12	115
	1313–1406	13	35
	1407–1455	18	49

The mean amplitude is 7·0 cm. and the phase at A.D. 641 is 100°.

(*f*) *Periodicity of 7·333 years.*

		a (cm.)	ϕ
A.D.	641– 728	12	157
	729– 816	16	140
	817– 904	9	138
	905– 992	20	275
	993–1080	20	174
	1081–1168	4	340
	1169–1256	17	152
	1257–1344	5	45
	1345–1432	19	184

The mean amplitude is 8·4 cm., the phase at A.D. 641 is 181°.

(*g*) *Periodicity of 8·33 years.*

		a (cm.)	ϕ
A.D.	641– 740	11	128
	741– 840	9	7
	841– 940	3	−6 (354)
	941–1040	19	−65 (295)
	1041–1140	4	60
	1141–1240	8	177
	1241–1340	7	102
	1341–1440	4	131

The mean amplitude over the whole period is only 1 cm. It will be noticed, however, that the phase angles show a regular variation with a length of about 500 years, the corresponding length of the periodicity varying from 8·1 to 8·5 years, the shorter length falling about A.D. 1100.

(*h*) *Periodicity of 11·06 years.*

This is very irregular in length, phase and amplitude:

		a (cm.)	ϕ
A.D.	641– 728	2	185
	729– 816	30	71
	817– 904	23	271
	905– 992	5	188
	993–1080	8	11
	1081–1168	11	237
	1169–1256	15	123
	1257–1344	7	246
	1345–1432	7	0

The mean amplitude is zero. This is due to a periodic variation of the phase having a length of about 500 years and a range of nearly 300 degrees, indicating a variation in length from about 10·86 to 11·36 years. The minimum length occurred about A.D. 900 and A.D. 1400, the maximum length about A.D. 1150. This is further discussed in section 8.

(*i*) *Periodicity of 12·50 years.*

		a (cm.)	ϕ
A.D.	641– 740	1	305
	741– 840	11	187
	841– 940	5	177
	941–1040	9	355
	1041–1140	11	169
	1141–1240	2	283
	1241–1340	3	233
	1341–1440	6	195

This periodicity is rather irregular and the amplitude is small, the mean amplitude over the whole period being only 3·2 cm., phase at A.D. 641, 194°.

(*k*) *Periodicity of 14·87 years.*

		a (cm.)	ϕ
A.D.	641– 715	18	125
	716– 790	11	167
	791– 865	13	337
	866– 940	14	74
	941–1015	20	6
	1016–1090	21	14
	1091–1165	12	42
	1166–1240	6	32
	1241–1315	12	97
	1316–1390	15	6
	1391–1451	27	133

This periodicity is well developed between A.D. 941 and about A.D. 1150, otherwise it is rather irregular. Over the whole period the mean amplitude is 8·0 cm., and the phase at A.D. 641 is 61°.

(*l*) *Periodicity of 16·68 years.*

A.D.	a (cm.)	ϕ
641– 739	12	331
740– 838	22	176
839– 937	16	24
938–1036	22	14
1037–1135	7	120
1136–1234	3	255
1235–1333	5	84
1334–1432	14	50

The mean amplitude is 5·0 cm., phase at A.D. 641, 42°.

(*m*) *Periodicity of 18·32 years.*

A.D.	a (cm.)	ϕ
641– 748	11	50
749– 856	20	108
857– 964	10	55
965–1072	9	122
1073–1180	4	61
1181–1288	6	199
1289–1396	10	33

The mean amplitude is 7·0 cm., phase at A.D. 641, 83°.

(*n*) *Periodicity of 22·12 years.*

This is fairly well developed.

A.D.	a (cm.)	ϕ
641– 728	12	169
729– 816	11	133
817– 904	8	5
995– 992	14	–20
993–1080	8	25
1081–1168	5	151
1169–1256	20	195
1257–1344	12	131
1345–1432	17	–41

If this is considered as a periodicity of uniform length, the mean amplitude is only 1·3 cm., but the periodicity has a very regular variation in length, the phase angle going through a complete wave with a range of about 220° in about 500 years (Fig. 3). The period accordingly varies from a maximum length of 22·43 years about A.D. 760 and A.D. 1260, to a minimum length of 21·81 years about A.D. 1010. (See also section 8.) When this variation is allowed for (see section 4 *d*), the amplitude is found to be 11·3 cm. The phase at A.D. 641 reduces to 68°.

(*p*) *Periodicity of 24·43 years.*

A.D.	a (cm.)	ϕ
641– 740	11	317
741– 840	13	296
841– 940	5	127
941–1040	13	35
1041–1140	5	175
1141–1240	6	157
1241–1340	19	32
1341–1440	30	304

This is rather irregular; the mean amplitude is 4·9 cm., and the phase at A.D. 641, 329°.

(*q*) *Periodicity of 33·49 years.*

A.D.	a (cm.)	ϕ
641– 739	10	29
740– 838	11	3
839– 937	13	359
938–1036	16	313
1037–1135	20	74
1136–1234	10	277
1235–1333	9	47
1334–1432	19	353

This periodicity is fairly regular, both in phase and amplitude. The mean amplitude is 9·4 cm., phase at A.D. 641, 5°.

(*r*) *Periodicity of 39·85 years.*

A.D.	a (cm.)	ϕ
641– 808	15	193
809– 976	6	159
977–1144	8	124
1145–1312	9	187
1313–1452	15	182

This periodicity is well developed with a mean amplitude of 9·7 cm., and phase at A.D. 641 of 176°.

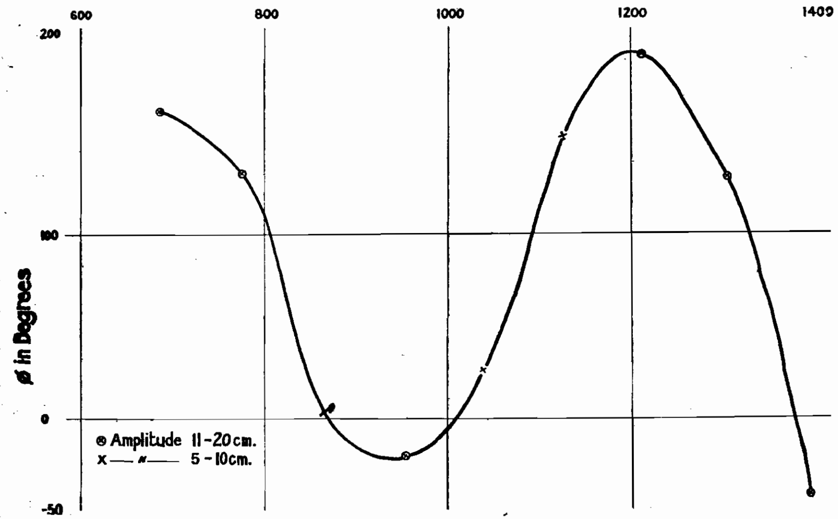

FIG. 3.—Systematic Variation of Phase, 22-Year Period.

(*s*) *Periodicity of 66·0 years.*

A.D.	a (cm.)	ϕ
641– 838	8	325
839–1036	13	325
1037–1234	15	28
1235–1432	5	282

The average length of the periodicity is 66·0 years, but it suffers a variation which completes a cycle in approximately 500 years. The range of the variation of ϕ is 106°, so that the length of the 66-year periodicity oscillates between 64·6 and 67·4 years, the shorter occurring about A.D. 1090, and the longer about A.D. 1340. The mean amplitude of the periodicity considered as uniform is 8·1 cm., and its phase at A.D. 641 is 343°. If allowance be made for the variation in length (see section 4 *d*), the mean amplitude becomes 10 cm.

(*t*) *Periodicity of 76·8 years.*

	a (cm.)	ϕ
A.D. 641– 800	20	37
801– 960	12	72
961–1120	14	53
1121–1280	15	57
1281–1440	26	42

This periodicity is well developed, and maintains an almost uniform phase. The mean amplitude is 16·9 cm., and the phase at A.D. 641 is 50°.

5. The Standard Deviation of the Nile Flood Levels.

For comparison with the amplitudes given in the preceding section the standard deviation of the data is required. Since the Nile flood levels have a secular increase owing to the deposit of silt, this could not be calculated directly in the ordinary way; the record has, therefore, been divided into sections of 100 years:

	cm.		cm.
A.D. 641– 739	68	1040–1139	47
740– 839	62	1140–1239	49
840– 939	53	1240–1339	46
940–1039	62	1340–1439	63

The average of the 800 years is 56 cm. The probable value of a periodic term of any given length arising by chance in a series of 800 random observations with this standard deviation is 2·7 cm. The odds against a periodicity with an amplitude of 8·1 cm. arising by chance in the determination of 19 different periodicities are even, and only the periodicity of 76·8 years could be regarded as real if the mathematical criterion was all there was to go on. The reality of the remaining periodicities depends on other considerations (sections 6–8). The problem of a criterion of reality is greatly complicated by the systematic variation of length and phase. The periodicity of 76·8 years with an amplitude of 16·9 cm. is almost certainly real.

6. The "Periodic Table."

As a result of the investigation we have obtained 19 periodicities of lengths varying from 1·91 to 76·8 years, some of them well established, others of doubtful validity, but all determined quite independently. It is, however, at once obvious that many of these periodicities bear a simple relation one to another, as, for example, the series:

5·52   11·06   22·12   33·49   66·0   76·8 years

which are approximately in the ratios:

½,   1,   2,   3,   6,   7.

Let us take as a basis the well-defined periodicity of 22·12 years, which is also a well-known cosmical periodicity. Then we can form a sort of periodic table of the sub-multiples and multiples of this periodicity (Table III.).

TABLE III.—The "Periodic Table."

Sub-Multiple of 22·12 years.	Length Years.	Periodicity found.	Ratio.	Multiple of 22·12 years.	Length Years.	Periodicity found.	Ratio.
Tenth . .	2·21	(2·177)	·985	1	22·12	22·12	1·000
Eighth . .	2·765	2·860	1·035	1½	33·18	33·49	1·009
Sixth . .	3·68	3·66	·995				
Fourth . .	5·53	5·52	·998	3	66·36	66·0	·995
Third . .	7·37	7·33	·995	3½	77·42	76·8	·992
Half . .	11·06	11·06	1·000				
Two-thirds .	14·75	14·87	1·009				
Three-fourths	16·59	16·68	1·005				
Five-sixths .	18·44	18·32	·994				

Leaving out of account the first two, we find that out of 16 periodicities with a length of more than 3 years, eleven agree within 1 per cent, with a simple multiple or sub-multiple of 22·12 years. This agreement is too striking to be the result of coincidence. In the first two periodicities the agreement is not so good, but that of 2·177 years could not be determined exactly because of the limitations of the method. On the other hand, the periodicity of 2·860 years is considered to be determined accurately to three decimal places, and in this instance the discrepancy of ·095 year or 3·5 per cent is too great. It therefore seems probable that this is not one of the 22-year group.

This leaves unaccounted for the periodicities of 6·83, 8·33, 12·50, 24·43, and 39·85 years. Of these 6·83 years is approximately five-sixteenths of 22·12, 8·33 years is almost exactly six-sixteenths, and 12·50 is approximately nine-sixteenths, but this apparent agreement may be accidental. These periodicities are all rather small and irregular; on the other hand all have traces of the curious 500-year cycle of length noted in the 22·12-year period. These may form another family which are harmonics of 354 years.

24·43 years is very close to one-fifth of 11 × 11·06 or 122 years, and rather curiously the other periodicity not accounted for, 39·85 years, is nearly one-third of this period. It also comes near one-ninth of 354 years. It was very difficult to determine the true length of this period from the drift of the phase angles, and it may represent a combination of $\frac{122}{3} = 40·7$ and $\frac{354}{9} = 39·3$ years. These, however, are rather forced analogies, and have little weight.

Mr. J. I. Craig, who saw the manuscript, also noticed this systematic arrangement—which in fact cannot well be missed—and arranged the periodicities in a form which brings out the relationships very clearly:

...	$\frac{P}{4}$	$\frac{P}{2}$	P	...
...	5·52	11·06	22·12	...
$\frac{P}{6}$	$\frac{P}{3}$	$\frac{2P}{3}$	...	...
3·66	7·33	14·87	...	...
...	$\frac{3P}{8}$	$\frac{3P}{4}$	$\frac{3P}{2}$	$3P$
...	8·33	16·68	33·49	66·0

Where P is 22·12 years.

accounting for ten of the periodicities. It seems best to regard 2·860 and 76·8 years as independent of the series, in spite of their close approximation to $P/8$ and $7P/2$.

7. Similar Periodicities in Recent Data.

(a) Solar Periodicities.

H. H. Turner[4] gives the following periodicities, resulting from a harmonic analysis of sunspots, which have an amplitude of more than 10 in the relative number.

Sunspots	78·0	52·0	12·0	11·14	10·40	9·75	8·67	8·21 years
Nile	76·8	...	12·5	11·06	...	...	...	8·33

The Nile flood periodicities which approach these more or less closely are added for comparison. The agreement, however, is not good. It must be remarked that the sunspot record is entirely subsequent to the Nile record and separated from it by 300 years.

The solar prominences are generally considered to have a periodicity of about 3·7 years (cf. 3·66 years). The ratio $\frac{\text{faculæ}}{\text{sunspots}}$ gives a periodicity of 2·2 years, which according to H. Arctowski[5] is of importance in terrestrial temperatures.

(b) The Brückner Cycle.

The length of this periodicity was determined by Brückner as $34{\cdot}8 \pm 0{\cdot}7$ years, but it is very variable. According to E. Huntington,[6] the "big trees" of western U.S.A. give a periodicity of 33·8 years. H. L. Moore,[7] found a periodicity of 33 years in the rainfall of the Ohio valley and Illinois. Moore gave his periodicity only to the nearest year; from his periodogram for the Ohio valley the true length is probably just over 33 years. Thus 33·49 years falls well within the range of the lengths variously given for this periodicity.

(c) D. Brunt's Periodicities in European Weather.

From periodogram analyses of various European long records,[8] Brunt determined 33 periodicities of over ten years distributed as follows:

Years	11	12	13	14	14½	15	16	17	17½	20	22	23	25	30	35
Number of periodicities	1	2	4	2	1	4	1	2	1	1	2	2	2	2	6

The lengths are very scattered, but there are indications of maxima at 12-13, 15 and 17 years, which may represent the 12·50, 14·87 and 16·68 year Nile flood periodicities.

In an earlier critical investigation of periodicities in Greenwich temperature, Brunt[9] found periodicities of approximately 1·92, 2·18, and

[4] On the expression of sunspot periodicity as a Fourier sequence. *London, Mon. Not. R. Astr. Soc.*, **73**, 1913, p. 549.
[5] The pleionian cycle of climatic fluctuations. *Proc. 2nd Pan-Amer. Sci. Congr.*, **2**, p. 172.
[6] "The climatic factor as illustrated in arid America." Washington, D.C., Carnegie Institution, 1914.
[7] "Economic cycles, their law and cause," New York, 1914.
[8] Periodicities in European weather. *London, Phil. Trans. R. Soc.*, A, **225**, 1925, p. 247.
[9] A periodogram analysis of the Greenwich temperature records. *London, Q.J.R. Meteor. Soc.*, **45**, 1919, p. 323.

2·92 years, which fit fairly closely with the Nile periodicities of 2·177 and 2·86 years, and suggest that of the two alternatives 1·91 and 2·10 years, the first is correct.

(d) Other Periodicities.

66·0 years.—Not previously recorded.

39·85 years.—Possibly represents a periodicity of 41 years in Greenwich pressure and temperature, and in H. H. Turner's rainfall "discontinuities."

24·43 years.—Not previously recorded.

18·32 years.—H. L. Moore's periodogram for Ohio valley rainfall shows a well-marked peak between 18 and 19 years, probably about 18·4 years. There is also a well-known lunar period of 18·6 years.

8·33 years.—An eight-year periodicity is known in a great number of meteorological records—winter pressure of Alps; Stockholm temperature; pressure, rainfall and floods of North America, Bathurst rainfall, tree-growth. It is often quoted as between seven and eight years, and may represent a combination of the 8·33 and 7·33 year periodicities. Brunt's careful analysis shows that periodicities between 8·2 years and 8·7 years are also frequent.

7·33 years.—F. Baur[10] finds a well-marked periodicity of 7·2 years in the pressure at a number of "action centres."

6·83 years.—Not previously recorded.

5·52 years.—The half sunspot-cycle, found in European rainfall and many other series.

8. The Cycle of Variation in the Lengths of the Periodicities.

I come now to the most difficult of all the many problems presented by the results of the investigation, namely the apparently systematic variation in length of the majority of the periodicities found. In the process of checking the results of the "difference-periodogram," the long record was divided into a number of sections of about 100 years, and the phases of the periodicity in each section, reduced to A.D. 641, were plotted to see if there was any progressive change of phase. If the phase decreases steadily, the trial period is too long, if the phase increases steadily, the trial period is too short. If during the 800 years covered by the data, the length of the periodicity remained constant, the phase angles of the different sections, when plotted, should lie along a straight line [it being remembered that a phase angle can be written indifferently as ϕ or as $(\phi \pm 360)$]. For the majority of the periodicities, however, it was found that these phase angles lay not along a straight line, but along a fairly regular curve, the best example being that for 22·12 years (Fig. 3). The curves approached sine curves with a length of about 500 years. The discovery was so unexpected—and incidentally so unorthodox[11]—that it called for further investigation.

The first step was to determine the length of this cycle of approximately 500 years more accurately. A preliminary investigation was carried out on the nine phase angles of the 22·12-year periodicity, for which were calculated the amplitudes of sine curves of lengths 420, 440, 460, 480, 500, 520, and 540 years. The curve representing these amplitudes showed *two* maxima, at about 450 and 520 years respectively.

[10] *Mitteilungen der Wetter- und Sonnenwarte St. Blasien, H.* 3, Braunschweig, 1924.
[11] But *cf.* H. W. Clough: "A systematically varying period with an average length of 28 months in weather and solar phenomena. Washington D.C.," *Monthly Weather Review*, **52**, 1924, p. 421.

Sine curves of these lengths were accordingly calculated for all the periodicities found.

TABLE IV.—VARIATION OF PHASE ANGLES.

		450-year cycle.			520-year cycle.		
Periodicity years.	σ_ϕ	a	θ	$\frac{a}{\sigma_\phi}$	a	θ	$\frac{a}{\sigma_\phi}$
	°	°	°		°	°	
2·860	26	27	333	1·04	24	0	·92
3·66	82	77	248	·94	125	79	·96
5·52	153	131	19	·86	146	101	·95
6·83	59	14	309	·24	35	228	·59
7·33	61	61	168	1·00	50	201	·82
8·33	92	104	6	1·13	125	43	1·36
11·06	125	146	359	1·17	146	4	1·17
	(152)	(201)	(350)	(1·33)	(174)	(30)	(1·15)
12·50	85	88	345	1·03	93	22	1·09
14·87	60	29	63	·48	32	10	·53
16·68	105	158	27	1·51	141	52	1·34
18·32	48	23	5	·48	6	260	··12
22·12	90	133	17	1·48	133	65	1·48
24·43	113	84	314	·74	129	354	1·14
33·49	47	46	38	·98	22	81	·47
39·85	24	19	331	·79	29	13	1·21
66·0	37	52	61	1·40	55	117	1·48
76·8	11	5	258	·45	7	212	·64
All periods (vector mean)	...	52	3	...	54	51	...

In Table IV. the first column shows the lengths of the different periodicities. The second column gives the standard deviations of the phase angles of the individual sections of the data. It must be remarked that in several instances the angle was regarded as $(\phi + 360°)$ or as $(\phi - 360°)$ instead of ϕ, the choice being governed by the run of the figures. For example, in the 22·12-year periodicity we have the following succession:

169° 133° 5° 340° 25° 151° etc.

The figure of 340° breaks the smooth run of the figures very abruptly, but if this figure be written down as −20°, the run is maintained. There can be no doubt that the latter interpretation is correct.

The only cases of real doubt occurred in the 11·06-year periodicity, where the oscillation of phase is especially great. In this case there were two doubtful values, that for A.D. 1169–1256, which may be either 123° or 483°, and that for A.D. 1345–1432, which may be either 0° or 360°. Two lines are given to this periodicity in Table IV.; in the first line these two doubtful figures have been omitted; in the second line (figures in brackets) they have been given the values (483° and 0°) which best fitted a smooth curve.

The third and fourth columns of Table IV. give the amplitude a and the phase θ of the variation in phase, in the expression:

$$(\phi - \bar{\phi}) = a \sin\left(\frac{2\pi t}{450} + \theta\right)$$

where $\bar{\phi}$ is the weighted average of the phase angle and t is the time in years since A.D. 641.

In all this part of the work the individual phase angles were weighted according to the corresponding amplitude. The fifth column gives the ratio between the standard deviations of the phase angles and the amplitude of the phase in the regular 450-year oscillation. This ratio forms a rough measure of the regularity of the variation of phase. The next three columns give the same particulars, the variation being regarded as having a length of 520 years.

It will be seen that of the 17 periodicities, for the 450-year cycle 13 values of θ fall in the angle of 110° bounded by 314·0° and 64·0°, while for the 520-year cycle 13 values fall in the angle of 124° bounded by 354·0° and 118·0°. Twelve of the periodicities fall in both these angles. In addition we find that the ratio a/σ_ϕ is in general much larger for the periodicities whose values of θ fall within the limits given above than for the remaining periodicities. It does not seem probable, however, that the variation is composed of two periodicities; it more probably consists of a single wave of about 500 years which does not follow a perfectly smooth sine curve.

It is not possible to say whether the variation of θ within the range of about 120° is real or accidental. Since all the periodicities seem to be related in some way to the 11 and 22 year periodicities, it would seem most consistent for all of them to reach their minimum and maximum lengths together. The scatter found would then be attributed to accidental variations, errors of the records and interference of the periodicities one with another.

The vector mean of all the different periodicities gives as the constants of the phase-variation:

$$450 \text{ year cycle}: (\phi - \bar{\phi}) = 52° \sin\left(\frac{2\pi t}{450} + 3°\right)$$

$$520 \text{ year cycle}: (\phi - \bar{\phi}) = 54° \sin\left(\frac{2\pi t}{520} + 51°\right).$$

It is difficult to say what is the real meaning of this apparently systematic variation of phase. It may be due to the interference of a major periodicity with a length of about 500 years, but there are several objections to this view. In the first place, the actual observations, corrected for secular variation, do not show any trace of the existence of such a periodicity. Secondly, the effects of the long periodicity should be almost completely eliminated in forming the difference-periodograms, but we have seen that in at least one instance, illustrated in Fig. 2, the variation in length is clearly shown on the difference-periodogram. Thirdly, the interference should give a well-marked regular variation of amplitude as well as of phase, but no clear indication of a regular variation of amplitude was found. The amplitudes are generally smallest in the twelfth and thirteenth centuries, and largest about A.D. 800 and 1400, but the variations are very irregular. Fourthly, the variation of phase caused by interference could not reach 180°, whereas in the 22-year period the range of variation is about 270°. Finally, interference could not reduce the resultant amplitude of a periodicity over the whole period of 800 years almost to zero, yet we find that the resultant amplitude of the well-marked 22-year period, if it is regarded as of constant length, is only 1·3 cm.

For all these reasons, it seems to me that the variation of phase of the different periods must represent a systematic variation in the true lengths of the periods. The length of a periodicity reaches its minimum where the phase is increasing most rapidly, *i.e.* 90° after the phase-minimum. Hence, on the 450-year cycle the periodicities would be shortest about A.D. 1087 and longest about A.D. 865 and A.D. 1312. On the 520-year cycle they would be shortest about A.D. 1088 and longest about A.D. 828 and A.D. 1348. The two dates of minimum length coincide, while the dates of maximum length are separated by nearly 40 years. This means that the minimum in the lengths of the periodicities is sharp, while the two maxima are diffuse. As to the significance of this peculiar variation in length, I do not feel competent to express an opinion.

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