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EDDY-VISCOSITY AND SKIN-FRICTION IN THE DYNAMICS OF WINDS AND OCEAN-CURRENTS

BY

V. WALFRID EKMAN, LUND.

Communicated by

Dr. L. F. Richardson, F.R.S.

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§ 1. Strictly, the very lowest layer of air in direct contact with the ground—or of water in contact with the sea-bottom—cannot have any velocity at all; the same applies to the sea-surface as far as the wind relative to the water is considered. So far there would, in the dynamics of winds and ocean-currents, be no room for the notion of skin-friction, but only for molecular viscosity and "eddy-viscosity," the compound effect of which may be expressed by a "virtual coefficient of viscosity" generally varying in the vertical direction.

From this point of view various researches have been made on the stationary wind or ocean-current produced by a pressure gradient varying with the height only, proportionally to the density of the medium.[1] For the sake of simplicity they will be here reported in terms applying to winds, though some of the researches were actually made with ocean-currents in view.

Allowing for turbulence and vertical displacements, the motion is completely represented by the curve, into which a vertical line becomes transformed when following for a unit of time the horizontal mean motion of the air. This curve will be called the "velocity curve."

§ 2. In the particular case when the kinematical virtual coefficient of viscosity is uniform over the whole space, the velocity curve was, by the present writer, found to be a three-dimensional equiangular spiral,[2] well known to English readers through the subsequent independent solution

[1] The present writer originally denoted such a current by the name of "gradient current." Subsequently, it has become customary in meteorology to apply the terms "gradient wind" and "gradient velocity" to the wind and wind's velocity which should take place in absence of friction, and which are actually attained at a sufficient height above the ground. Adopting this convenient notation in oceanography as well, the term gradient current should be reserved for the hypothetical stationary motion produced by the pressure gradient in absence of friction. I have not, however, been able to find another suitable English word (like the German "Staustrom") for the entire current produced by the pressure gradient, and including as well the lower layers which are checked by friction; and I would be obliged for any useful suggestion. The notion is one of those indispensable in dynamical oceanography.

[2] V. Walfrid Ekman, "On the Influence of the Earth's Rotation on Ocean-Currents," *Ark. Matem., Stockholm*, **2**, 1905, No. 11.

of G. I. Taylor.[3] The spiral is of a constant pitch, each half-turn corresponding with a layer of air of the thickness

$$D=\pi\sqrt{\frac{\mu}{\rho\omega \sin\phi}}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ denotes the density, μ the virtual coefficient of viscosity, ω the angular velocity of the earth, and ϕ the latitude. This quantity D (denoted in Prof. Taylor's paper by π/B) is proved to be—whether uniform or varying with the height—of fundamental importance in the dynamics of ocean-currents. It denotes in a way the greatest vertical distance within which horizontal motion can be transferred by friction, and on that account I have called it the "Depth of Frictional Influence," having, unfortunately, not been able to find in English a shorter and handier term.[4]

If ρ varies considerably with the height z, as may be the case in dynamical meteorology, it will be convenient to introduce ρD instead of D as a characteristic of the amount of friction, and at the same time to replace z as independent variable by $\int\rho dz$; but that will not be done in the present communication.

§ 3. Returning to the question of the velocity curve under the assumptions of the preceding article, let u and v be the components of velocity relative to the gradient wind or gradient current, the direction of v taken at a right angle *contra solem*[5] from the direction of u, which latter direction may be chosen arbitrarily. Let the vertical co-ordinate z be reckoned: in the case of winds upwards from the ground or from the sea-surface; in the case of ocean-currents in the neighbourhood of the sea-bottom upwards from the latter; and in the case of drift-currents downwards from the sea-surface. The relation between u, v and z is then given by the equations

$$u=V_0e^{-\frac{\pi z}{D}}\cos\left(\epsilon-\frac{\pi z}{D}\right);\ v=V_0e^{-\frac{\pi z}{D}}\sin\left(\epsilon-\frac{\pi z}{D}\right), \quad . \quad (2)$$

V_0 and ϵ being arbitrary constants.[6] On account of these equations, which will be referred to presently, the vector of du/dz and dv/dz everywhere forms an angle of 45° with the vector of u and v; and the angle between the gradient wind and the surface wind (*i.e.* the wind very near the ground) should therefore be 45°.

§ 4. This latter result is not in good agreement with actual observations, even if it be considered that the observations on "surface wind" actually refer to the wind some metres above the ground. To improve the agreement materially, it is necessary to consider the variation of μ with height, μ being much reduced in the neighbourhood of the ground (or of the sea-surface or the sea-bottom, respectively).

It is easily seen[7] that even in case of μ variable with height the velocity curve will be a three-dimensional spiral, winding round the vertical line that represents the gradient wind, and asymptotically approaching it with increasing z. But it is a matter of considerable

[3] G. I. Taylor, "Eddy Motion in the Atmosphere," *London, Phil. Trans. R. Soc.*, A, **215**, 1914, p. 1.

[4] Here also I should be obliged for any good suggestion. The German translation is "Reibungstiefe."

[5] The terms *cum sole* and *contra solem* indicate the two directions of rotation usually characterised in meteorology as anticyclonic and cyclonic.

[6] The approximate validity of this solution requires, in the case of ocean-currents, that the depth of the ocean is great as compared with D, or, in any case, not smaller than D.

[7] V. Walfrid Ekman, "Beiträge zur Theorie der Meeresströmungen," *Ann. Hydrogr., Berlin*, **34**, 1906.

difficulty to find, even approximately, the actual laws of variation of μ, and from them to determine the shape of the spiral. Endeavours which have been made to this end have not been very successful. Thus, a calculation made on the assumption of μ proportional to the velocity of shear of the mean motion,[8] came out a failure, since this assumption implied an *increase* instead of a decrease of μ towards the ground (or the sea-surface).

§ 5. A recent calculation of H. Solberg[9] is of more interest in this connection. It is based on the assumption of ρ constant and μ increasing with z according to the formula

$$\mu = a(z+b)^2, \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

a and b being constants. The geometrical difference $\surd(u^2+v^2)$ between the gradient wind and the wind at the height z is, on this assumption, found to decrease upwards inversely as

$$\left(1+\frac{z}{b}\right)^{m+\frac{1}{2}},$$

and at the same time to veer through the angle

$$n \cdot \log_e\left(1+\frac{z}{b}\right),$$

m and n denoting the positive square roots of

$$m^2 = \frac{1}{8}+\sqrt{\frac{1}{64}+\frac{\rho^2\omega^2\sin^2\phi}{4a^2}}\,; \quad n^2 = -\frac{1}{8}+\sqrt{\frac{1}{64}+\frac{\rho^2\omega^2\sin^2\phi}{4a^2}}.$$

The angle α between the gradient wind and the surface wind (for z infinitesimal) is

$$\alpha = \text{arc tan}\,\frac{2n}{2m+1},$$

and can be written as a function of the derivative dD/dz, decreasing from 45° to 0, when dD/dz increases from 0 to ∞.

§ 6. There is, of course, from a *physical* point of view, no particular reason for adopting the assumption (3), which happens to make the mathematical problem treatable; and its value depends entirely on the degree of accuracy with which the wind-velocities and directions calculated from it may be made to fit the results of actual observations. Owing to the constants a and b, Solberg's solution can be made to fit two given conditions, as *e.g.* any given values of α (short of 45°) and of the height in which the wind-direction coincides with the direction of the gradient wind; but, nevertheless, it would not represent very closely an actual field of wind. For the rate of variation of the wind with the height comes out comparatively too slow in the higher as well as in the very lowest layers, and too rapid in the intermediate ones; and, furthermore, with actually occurring values of α, the solution does not give the characteristic increase of velocity and veering of wind sensibly *beyond* the velocity and the direction of the gradient wind. And—as a consequence of the former circumstance—the ratio between the heights at which this velocity and this direction are attained, comes out much too small.

To exemplify this we will consider some pilot balloon observations of

[8] "On the Influence of the Earth's Rotation on Ocean-Currents," *loc. cit.*, p. 43.

[9] H. Solberg, "Sur le frottement dans les couches basses de l'atmosphère," *Förhandl. vid det* 17 *de skandinav. naturforskarmötet* (*Göteborg*, 1923), printed in Göteborg, 1924.

G. M. B. Dobson,[10] quoted in G. I. Taylor's paper (*loc. cit.*). With the latter author we will use particularly the averages for "strong winds" (above 13 m./s.), the averages calculated for "light" and "moderate" winds being less reconcilable with Solberg's or any other dynamical solution. In Fig. 1 the cross-marks show for the level of the anemograph (unknown to the present writer), and for 7 other heights z the ratio r between the wind-velocity and the gradient velocity taken from the tables in the original paper. In Fig. 2 the cross-marks show similarly, for 4 heights the angle ψ between the directions of wind and gradient wind. (The change of sign of this angle at about 1000 or 1100 m. is not seen in the figure, since the diagram has been drawn up to 900 m. only.) In the same figures the three full-drawn lines 1, 2, 3, represent three various

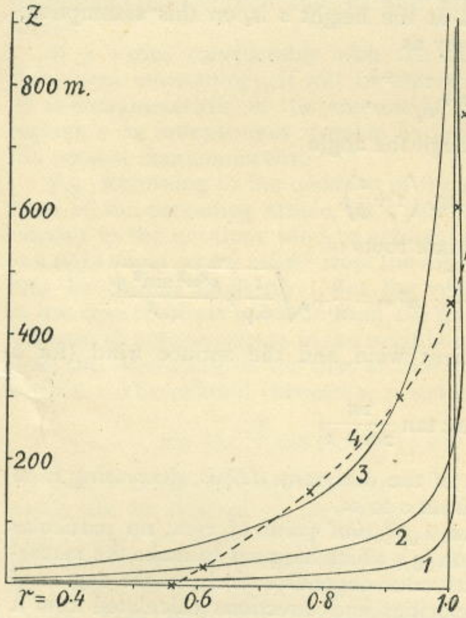

Fig. 1.

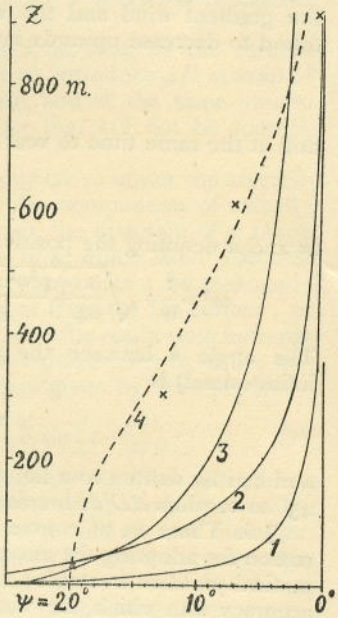

Fig. 2.

attempts to adjust Solberg's solution according to the values given by the cross-marks. The curves 1 are drawn so that $\psi = 0$ at 1100 m. and $\psi = 20°$ at 6 m. above the ground. (Close to the ground $\psi = \alpha = 27°{\cdot}2$.) The curves 2 are drawn similarly, only that the value $\psi = 20°$ is attained at 30 m. from the ground (and close to the ground $\psi = \alpha = 32°$). These curves 1 and 2 obviously do not agree at all with the observations; within a large interval of altitudes, reckoned from some 50 metres upwards, they give much too large velocities and much too small values of ψ. Finally, the curves 3 are drawn so as to make the ratio r agree with the observed velocity at 300 m., and so that $\psi = 19°$ at 30 m. (Close to the ground $\psi = \alpha = 23°{\cdot}7$.) In addition to other discrepancies, it is to be noted that these curves give much too small velocities in the neighbourhood of the ground, whereas the heights at which the gradient velocity

[10] G. M. B. Dobson, "Pilot Balloon Ascents at the Central Flying School, Upavon, during the year 1913," *Q.J.R. Meteor. Soc.*, **40**, 1914, p. 123.

and the gradient wind direction are attained, come out much too large (about 1500 m. and 25,000 m. respectively).

To illustrate further the three particular solutions here discussed, the horizontal projections of the three corresponding spirals are given as full-drawn curves: 1, 2 and 3 in Fig. 3 below. No comment is necessary, except that the curves, in order to be consistent with the observations in question, should pass through the 4 cross-marks.

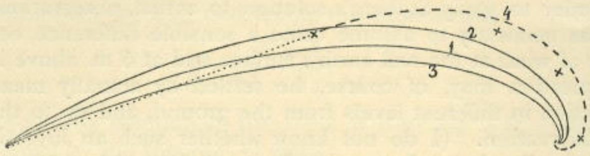

FIG. 3.

§ 7. There is another simple way of reconciling the theory with the observed values of α, namely, by disregarding the lowest layer of air—we may call it the "skin-layer"—in which eddy-viscosity is much reduced, and by calculating with a uniform coefficient of viscosity for the rest of the air, which we assume to move at a finite velocity over a fictitious ground, experiencing there a stress straight opposite to the direction of motion. The velocity curve is even then an equiangular spiral of uniform pitch (represented in horizontal projection by AB, Fig. 4), but the origin at the ground must be taken outside the spiral in a point O on the tangent OAT.

I suggested the method in 1906,[11] and pointed out that in this way any angle α between zero and 45° (or up to 49°·1 in case of the "quadratic" law of friction alluded to in § 4 above) may be accounted for, depending on the velocity V_s of the surface wind as compared with the velocity G of the gradient wind. It is the same method that was independently applied and carried out with so much success by G. I. Taylor, who, in his repeatedly quoted paper on "Eddy Motion in the Atmosphere," showed that the solution in question could be made to fit very well Dr. Dobson's observations on "strong winds." It is, for comparison with Dr. Solberg's solution, represented in Figs. 1-3 by the broken lines 4, extended in Fig. 3 by a dotted line corresponding with the skin-layer.

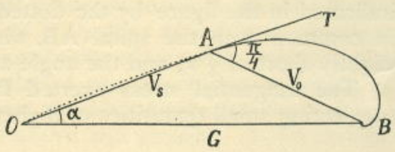

FIG. 4.

The important relation $V_s = G(\cos\alpha - \sin\alpha)$ given in Prof. Taylor's paper (equation 17, p. 16) is on the above-mentioned premises an immediate consequence of the theorem of sines, since in the triangle OAB in Fig. 4 the sides V_s and G are opposite the angles $\frac{1}{4}\pi - \alpha$ and $\frac{3}{4}\pi$.

§ 8. The method set forth in the preceding section rests on the assumption that the velocity over the fictitious ground is in the direction of stress at the same level, which implies that the mean motion has, in spite of the earth's rotation, the same direction in all levels within the skin-layer. This assumption, though it may appear rather obvious, perhaps needs justification. For it implies that the thickness δ of the "skin-layer" is a small fraction of the average "depth of frictional

[11] "Beiträge zur Theorie Meeresströmungen," p. 484, *loc. cit.*

influence" D_s within the same layer, and, on the other hand, it will be shown that the observed wind-velocities necessarily require a certain magnitude of δ as compared to D_s.

It is not evident that this condition is actually satisfied. As a matter of fact, the quantity D_s might theoretically—for complete absence of eddy-viscosity—be even smaller than the ordinary heights of observation, being at the poles something about 140 cm. according to temperature, and about twice as great at a latitude of 15°. It will also be remembered that, in order to apply Solberg's solution to actual observations of wind (§ 6), it was necessary to assume quite a sensible difference between the directions of wind at the real earth's surface and at 6 m. above it.

The question may, of course, be settled by actually measuring the wind direction in different levels from the ground, and up to the ordinary level of observation. (I do not know whether such an investigation has been made.) But it can be answered as well from the measured value of α combined with observations of the wind in the upper air.

§ 9. For this purpose we will assume, below the level $z=\delta$, a uniform coefficient of viscosity μ_s and, correspondingly, a uniform depth of frictional influence D_s. Above that level there should be another uniform and much greater coefficient of viscosity μ and a corresponding uniform value D. Furthermore, we will assume that the condition required is, approximately, fulfilled, *i.e.* that δ/D_s is a small fraction, and we shall calculate *how small* it is.

We may still keep to Fig. 4, but with the alteration that OAT is not a straight line.[12] In the skin-layer there will be a real velocity *curve* indicated in the figure by the dotted curve OA, and with the tangent AT in common with the spiral AB, whence the obtuse angle OAB will be slightly short of $\frac{3}{4}\pi$, and the angle at B slightly greater than $\frac{1}{4}\pi-\alpha$.

The tangential stress exerted from above on the skin-layer is then (neglecting small quantities of higher order)

$$T=\frac{\mu_s V_s}{\delta}.$$

Solving this equation for δ, and substituting for μ_s the value which is obtained from (1) if D be replaced by D_s, we find

$$\delta=\frac{\rho\omega\sin\phi\cdot V_s D_s^2}{\pi^2 T}. \qquad (4)$$

On the other hand, the upper air must experience from below the same tangential stress T, so that, if in the following the vertical co-ordinate z be reckoned from the common boundary between the skin-layer and the upper air,

$$T=\mu\left[\sqrt{\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2}\right]_{z=0}.$$

In this equation we replace u and v by the expressions (2) and solve with regard to V_0, which now denotes the magnitude of the vector difference between the gradient wind and the "surface wind." On account of (1) we find in this way

[12] Fig. 4 is drawn slightly false, with the intention that it may illustrate at the same time either of the two incompatible assumptions made in §§ 7 and 9. The full-drawn line OA should, in § 7, represent a tangent to the curve AB, but represents in § 9 a secant in the curve OAB the tangent of which is AT.

$$V_0 = \frac{TD}{\sqrt{2}\cdot\pi\mu} = \frac{\pi T}{\sqrt{2}\cdot D\rho\omega\sin\phi}. \qquad (5)$$

Again, if we write m for the ratio V_s/V_0, and if in (4) we replace V_s by mV_0 and eliminate V_0 by means of (5), we get

$$\frac{\delta}{D_s} = \frac{mD_s}{\pi\sqrt{2}\cdot D}, \qquad (6)$$

so, after multiplying by δ/D_s and taking the square root,

$$\frac{\delta}{D_s} = \sqrt{\frac{m}{\pi\sqrt{2}}}\cdot\sqrt{\frac{\delta}{D}}. \qquad (7)$$

Now, as already mentioned, the angle at B in Fig. 4 is, approximately, $45^\circ - \alpha$, whence by the theorem of sines

$$m = \frac{V_s}{V_0} = \frac{\cos\alpha - \sin\alpha}{\sqrt{2}\cdot\sin\alpha}, \text{ and } \frac{m}{\pi\sqrt{2}} = \frac{\cot\alpha - 1}{2\pi}, \qquad (8)$$

approximately. This latter quantity decreases from 0·97 for $\alpha = 8^\circ$ to 0·43 for $\alpha = 15^\circ$ and to 0·18 for $\alpha = 25^\circ$, so that—since values of α smaller than 8° are hardly known to exist—one may write for (6) and (7)

$$\frac{\delta}{D_s} < \frac{D_s}{D} \text{ and } \frac{\delta}{D_s} < \sqrt{\frac{\delta}{D}}.$$

Of these two inequalities the second one is particularly worth noticing. Considering the very limited degree of accuracy to which the theory can be realised for the present, any value of δ/D_s short of 0·1, and, therefore, any value of δ/D short of 0·01, will justify the assumption mentioned at the top of § 8. The question is whether δ can be chosen so small, *i.e.* whether the laws of motion—empirical or theoretical—for the upper air are valid down to the height 0·01 D above the ground. If, on account of the data given in Prof. Taylor's paper, p. 21, we assume D to exceed 600 m. on land and 100 m. on the sea,[13] it would be sufficient, therefore, if the laws of motion of the upper air were applicable down to 6 m. above land and 1 m. above the sea. This condition is probably fulfilled, and the notion of skin-friction should therefore be allowable. The subject might, however, deserve a fuller quantitative examination.

§ 10. As for the magnitude of the skin-friction, G. I. Taylor found[14] that it could be written in the form

$$T = \kappa\rho V_s^2, \qquad (9)$$

where κ is a dimensionless constant, estimated over Salisbury Plain to be about 0·002 or 0·003. A rough calculation[15] made for the air above the Baltic during the storm in Nov. 1872 gives—when expressed in our present notation—the identical value $\kappa = 0{\cdot}0025$.

We shall not dwell here on the possible—and probable—dependence of κ on the nature of the earth's surface, but only on the circumstance that κ is independent of the latitude, whereas D increases towards the

[13] Prof. Taylor's quantity H_1 is $D\left(\frac{3}{4}+\frac{\alpha}{\pi}\right)$, so that with the actual values of α, D will come out some 15 or 20 per cent greater than H_1.

[14] G. I. Taylor, "Skin Friction of the Wind on the Earth's Surface," *Proc. R. Soc., London*, A, **92**, 1916, p. 196.

[15] V. Walfrid Ekman, "On the Influence of the Earth's Rotation on Ocean-Currents," *loc. cit.*, p. 40.

equator. This circumstance suggests, as will be shown in the next three paragraphs, an interesting conclusion with regard to the direction of the surface wind.

§ 11. From the first equation (5) it appears that $\mu V_0/D$ is a function of T alone and independent of ϕ. Now imagine two localities in different latitudes, and suppose that the value of T, as well as the nature of the ground's surface, are equal at both places. Since $\mu V_0/D$ has then the same value at both places, and since μ is not likely to decrease with increasing intensity of relative motion, *i.e.* with increasing V_0/D, it should be expected that μ and V_0/D are *each* independent of ϕ and functions of T alone.

Theoretically this latter conclusion is certainly not unobjectionable, since an increase of the linear dimension D may cause an increase of eddy-viscosity. But it is supported by investigations on ocean-currents. Empirical investigations [16] show that the velocity of the currents varies inversely as $\sqrt{\sin\phi}$, wind and other circumstances supposed to be unaltered. From this law, and from the proportionality between the velocities of wind and current and between the latter and the square root of T, it follows (*vide* § 16) that the ratio V_0'/D' between the velocity of the pure drift-current and the depth of frictional influence is independent of T as well as of ϕ, whence—on account of (5)—μ is independent of ϕ. By analogy the same, therefore, should be expected in the atmosphere below the region of the gradient wind.

Writing, therefore, $D = V_0 \cdot F(T)$, and substituting this expression for D on the right-hand side of (5), we have

$$V_0^2 \cdot F(T) = \frac{\pi T}{\sqrt{2} \cdot \rho\omega \sin\phi}. \quad . \quad . \quad . \quad (10)$$

§ 12. The real relationship between V_0 and T is, as far as I know, not yet ascertained; but on account of (9) it would be implied in the relationship between any two of the quantities V_s, V_0, G, and α, determining the triangle in Fig. 4. From Dr. Dobson's observations on V_s/G for "light," "moderate" and "strong" winds, Prof. Taylor calculated ("Eddy Motion in the Atmosphere," p. 17) that the angle α should increase with increasing velocity of the wind, whence V_0/V_s as well should increase with T. On the other hand, the observed angles α were slightly greater in the case of moderate winds than in the case of strong winds, so that the aforesaid observations do not seem to be decisive on the point in question. I am not sufficiently acquainted with meteorological literature to know whether the relationship between V_0 and V_s has otherwise been settled, but some suggestion could be got from the laws of drift-currents in the ocean. For owing to the small velocities of the currents as compared with that of the wind itself, the velocity V_s of the surface wind determines the tangential stress exerted as well on the surface water as, from below, on the air; and since the two components of the velocity V_s' of the surface water, *i.e.* the gradient velocity and the velocity V_0' of the pure drift-current, are supposed to increase with increasing wind, in the same proportion (*vide* § 16), one should expect similar relations to hold between V_s' and V_s, between V_0' and V_s, and between V_0 and V_s.

From observations on lightships, chiefly in the inner Baltic, Witting

[16] H. Thorade, "Die Geschwindigkeit von Triftströmungen und die Ekmansche Theorie," *Ann. Hydrogr., Berlin*, **42**, 1914, p. 379.

C. S. Durst, "The Relationship between Current and Wind," *Q.J.R. Meteor. Soc.*, **50**, 1924, p. 113.

concluded[17] that V_s' should increase at a *slower* rate than V_s, or approximately as $\sqrt{V_s}$. Thorade (*loc. cit.*) applied a correction to this calculation by using Köppen's values for converting Beaufort forces into metres per second; with the result that for wind velocities above 5 or 6 m./sec. V_s' came out approximately proportional to V_s, whereas for weaker winds V_s' came out proportional to $\sqrt{V_s}$. On the other hand, a large number of observations on the Adlergrund lightship, midway between Rügen and Bornholm, gave, according to Dr. Thorade (*loc. cit.*), a direct and accurate proportionality between wind and current for all values of V_s between 4·4 and 17·4 m./sec. More recently C. S. Durst[18] found, by an analysis of some 2000 observations from the three great oceans, an almost true proportionality between V_s' and V_s for wind velocities between 2 and 12 m./sec. This latter investigation, as well as the Adlergrund investigation, both satisfy the two valuable conditions that the observations belong to the open sea, and that comparison was not made between statistical mean values, but between individual observations of wind and current. I think, therefore, there is good reason to regard V_s' as proportional to V_s, and by analogy to assume V_0/V_s and α to be independent of the strength of the wind.[19] On account of (9), V_0^2 would then be proportional to T, and the function $F(T)$ in (10) would be a constant.

§ 13. If this be true, we should find by elimination of T between (9) and (10) and considering (8) that V_s/V_0 and $\cot \alpha - 1$ are proportional to $\sqrt{\sin \phi}$. The angle α should therefore increase from the poles towards the equator, where it should be 45°.

When looking for confirmation of this result, I was surprised to find in the literature very little information on the alteration of α with latitude. Loomis has a series of data showing a regular increase of α from the poles towards the equator; but ranging from 28°·6 in the Arctic regions to 62°·2 on the Philippine Islands, they differ considerably from the results of more modern determinations. On the other hand, Hann quotes in his "Meteorologie" an interesting account of Blanford on tropical cyclones.[20] From some 200 observations on the open sea or at coast stations, Blanford computes the average value of α to be 32° or 33° at latitudes between 22° and 15° and 39° at latitudes between 15° and 8°. According to Kassner, the corresponding average *without* regard to latitude should be 18° for coastal stations and 10° on the sea. In conclusion, there is apparently an increase of α towards the equator, and though this *might* to some extent be explained by the circumstance that near the equator the conditions are less favourable for the development of stationary motion, yet the agreement with the theory may deserve attention.

The following table gives the values of α which, according to the theory, would be expected at various latitudes under conditions (nature of the earth's surface, etc.) which, at the poles, would make $\alpha = 10°, 15°, 20°, 25°$ or $30°$:

[17] R. Witting, "Zur Kenntnis des vom Winde erregten Oberflächenstromes," *Ann. Hydrogr., Berlin*, **37**, 1909, p. 193.

An investigation by E. Koch on wind and current in the North Atlantic (*ibid.* **51**, 1923, p. 201) gave a similar result.

[18] C. S. Durst, "The Relationship between Current and Wind," *Q.J.R. Meteor. Soc.*, **50**, 1924, p. 113.

[19] This conclusion is not in disagreement with a statistical investigation by H. Jeffreys (*London, Proc. R. Soc.*, A, **96**, 1919, p. 233) of some 600 wind variations over the North Sea. For when calculating from the Tables I. and II., p. 239 of his paper, the average values of α and V_s/G (S/G in Dr. Jeffreys' notation) for different values of G, there seems to be no reliable correlation to be found between α and G or between V_s/G and G.

[20] *Nature, London*, **38**, 1888, p. 182.

COMPUTED VARIATION OF α WITH LATITUDE.

$\phi=$	0°	2°	5°	10°	20°	40°	60°	90°
$\alpha=$	45°	28°·1	22°·8	18°·7	15°·0	11°·9	10°·6	10°·0
	45°	33°·5	29°·0	25°·1	21°·1	17°·4	15°·8	15°·0
	45°	37°·0	33°·4	30°·1	26°·3	22°·6	20°·8	20°·0
	45°	39°·5	36°·8	34°·1	30°·9	27°·5	25°·8	25°·0
	45°	41°·3	39°·4	37°·5	35°·0	32°·2	30°·7	30°·0

As far as the tabulated effect is concerned, it is seen that a considerable increase of α takes place chiefly at very low latitudes.

§ 14. Turning now to the problem of ocean-currents, there seems to be good reason even here to introduce the assumption of slipping at the bottom, though it cannot be supported by actual observations. A quantitative calculation founded on this assumption was for the first time carried out by H. Jeffreys in an interesting paper[21] of 1923. The author starts from the properties of the pure drift-current, *i.e.* the current produced by the wind blowing over a horizontal sea-surface; and he makes it his object to investigate how such a current would be modified—by the addition of a pressure gradient—outside a long straight coast which prevents any resultant drift from taking place across a vertical section parallel to the coast. The very same problem has, by the present writer, been made the subject for a similar investigation,[22] though with the important difference that the assumption of slipping at the sea-bottom was not introduced.

For the skin-friction at the bottom Dr. Jeffreys adopts the same law (equation 9) that was introduced by G. I. Taylor in the theory of winds. For the coefficient κ entering into this equation he uses—without explicit argument—the value $\kappa = 0{\cdot}002$. This is in good agreement with the values mentioned above in connection with the winds, and it is interesting to note that it agrees as well with experiments with water in tubes and channels. Owing to the big difference of densities in the one case and of linear dimensions in the other, as well as of the nature of the surfaces, it would be very desirable to have the said value tested by actual observations in the sea. Until such an investigation is possible, there is in any case good reason to use the value $\kappa = 0{\cdot}002$ for a trial.

§ 15. On the other hand, I am less willing to accept Dr. Jeffreys' assumption of a fixed value $k = 100$ C.G.S. of the kinematical virtual coefficient of viscosity $k = \mu/\rho$. It would imply that, with increasing intensity of the motion, the tangential stress should increase in *linear* proportion to the velocity of shear of the water everywhere except in the bottom layer or "skin-layer" where, according to (9), it would increase as the *square* of the velocity of shear.

This assumption appears rather unlikely in itself,[23] and it is not confirmed by its consequences, which are clearly laid down and emphasised in Dr. Jeffreys' paper. With increasing wind the current would increase at a proportionally *higher* rate than the wind itself; the relation being exactly a linear one only in the limiting case of very weak winds or winds

[21] H. Jeffreys, "On the Effect of a Steady Wind on the Sea-Level near a Straight Shore," *Phil. Mag.*, *London*, **46**, 1923, p. 114.

[22] "On the Influence of the Earth's Rotation on Ocean-Currents," *loc. cit.*, pp. 19-37.

[23] In a previous paper (*Phil. Mag.*, *London*, **39**, 1920, p. 578) Dr. Jeffreys has mentioned various circumstances likely to cause a variability of k. What I wish here to point out as essential is, however, the necessity in some way or other of considering this coefficient as a function of the intensity of the mean motion itself.

directed nearly at right angles to the shore, whereas in the other limiting case of a "hurricane alongshore" the velocity of the current would be in proportion to the *square* of the velocity of the wind. Parallel with this transition from one law of velocity to another, the angle α would gradually increase from zero to 45°. This latter result cannot be confronted yet with experience. But as to the alleged increase of the current-velocity at a higher rate than the wind, it is decidedly at variance with observations, these indicating, as already mentioned, *either* a direct proportionality between wind and current, *or* a *slower* rate of increase of the latter.

§ 16. From the relationship between wind and current it will be possible to come to a conclusion about the laws according to which the depth of frictional influence and the virtual coefficient of viscosity increase with the velocity of the current.

When for this purpose we consider the motion of the "hydratmosphere," *i.e.* the sea and the air above it, the velocity curve will show three separate spirals occupying the lower part of the air, the surface layer of water and the bottom layer of water respectively. Corresponding to these spirals we have three different depths of frictional influence, which may be denoted by D, D' and D'', and which are proportional to the square roots of three virtual kinematical coefficients of viscosity μ/ρ, μ'/ρ' and μ''/ρ''. Furthermore, we have three surface velocities: V_s in the air at the top of the skin-layer at the surface of the sea, V_s' in the surface water at the bottom of the same skin-layer, and V_s'' in the water at the top of the skin-layer at the sea-bottom (Fig. 5). We have also two gradient velocities: G in the air and G' in the water (provided the air and the water are homogeneous); and finally, three vector differences of velocity corresponding to the three spirals, namely, $V_0 = G - V_s$ in the air, the velocity of the "pure drift-current" $V_0' = V_s' - G'$ in the surface-water, and $V_0'' = G' - V_s''$ in the bottom water.

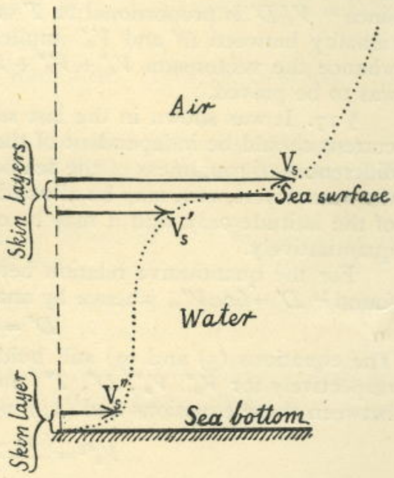

FIG. 5.—Scheme of notation, drawn for simplicity as if all the velocities were parallel, although they need not be so in fact.

On a previous occasion, when no slipping at the sea-bottom was assumed, I concluded [24] from the proportionality between wind and current (*i.e.* between V_s and V_s') now vindicated in § 12, that D' should be proportional to V_0', and by analogy D'' proportional to G'. In case of slipping the same reasoning would have led to the proportionality between D' and V_0' and between D'' and V_0''. It may be sufficient for the present to show that, in a sea of uniform depth and lying within a very narrow range of latitudes, these latter laws are consistent with—and in fact have for consequence—the proportionality between V_s' and V_s. (If regard has to be paid to the topography of the sea-bottom and to the differences of latitude, the results will come out more complicated.)

[24] "Über Horizontalzirkulation bei winderzeugten Meeresströmungen," *Ark. Matem., Stockholm*, **17**, 1923, No. 26.

The proportionality between D'' and $V_o''^2$ implies that the velocity of shear on top of the skin-layer at the sea-bottom is independent of the velocity of the current, whence the tangential bottom stress T''' is proportional to μ'', *i.e.* to D''^2 or to $V_o''^2$. On account of the law of skin-friction (9) therefore V_o'' is proportional to V_s'', whence the angle α'' of the bottom current is invariable. This implies further, since the thickness of the skin-layer is assumed to be insignificant, that the quantities of water S_y'' and S_x'' carried in unit of time by the bottom current severally in the direction of the gradient and at right angles to it, are both proportional to $V_o''D''$, *i.e.* to $V_o''^2$. But I have shown [24] that the pressure gradient and the gradient velocity in the ocean are determined—as far as differences of depth and of latitude could be left out of consideration—by the condition that these quantities S_y'' and S_x'' should balance the quantities of water carried to or from any region of the sea by the pure drift-current, and that S_y'' and S_x'' should therefore be proportional to T, that is, to V_s^2; whence V_o'' and V_s'' are proportional to V_s. Furthermore, since [25] $V_o'D'$ is proportional to T or to V_s^2 (see equation 9), the proportionality between D' and V_o' implies that V_o' too is proportional to V_s; whence the vector-sum $V_s'' + V_o'' + V_o' = V_s'$ is proportional to V_s, which was to be proved.

§ 17. It was shown in the last section that the angle α'' of the bottom current should be independent of the velocity of the current. In as far as differences of roughness of the sea-bottom and of the degree of stability of the water-layers, etc., may be allowed for, α'' should therefore be a function of the latitude only, and it may be of interest to work out this connection quantitatively.

For the quantitative relation between D' and V_o' alluded to in § 16, I found [24] $D' = 600\, V'_o$, whence by analogy we may assume for a trial:

$$D'' = 600\, V_o''. \qquad . \quad . \quad . \quad . \qquad (11)$$

The equations (5) and (9) still hold, if V_o, V_s, D, T and ρ be exchanged respectively for V_o'', V_s'', D'', T''' and ρ''. By elimination of D'' and T''' between these equations and (11), we find

$$V_s''^2 = \frac{600\sqrt{2}\cdot\omega\sin\phi}{\pi\kappa} V_o''^2. \qquad . \quad . \quad . \qquad (12)$$

Hence we may, for a given value of κ, calculate numerically V_s''/V_o'', and V_s''/G' and, on account of the equation corresponding to (8), α'' as well. For $\kappa = 0{\cdot}002$, α'' is found to be $10^\circ{\cdot}4$ at the poles, so that at the various latitudes given in the table, p. 170, α'' should be slightly greater than the corresponding uppermost values of α. V_s''/G' would at the equator and at the latitudes of 2°, 5°, 10°, 40° and 90°, be 0, 0·39, 0·52, 0·61, 0·78 and 0·80 respectively.

This calculation is of course very uncertain, chiefly owing to the uncertainty of the value of the coefficient κ in (9) and of the numerical coefficient in (11). On the other hand, if it were possible by actual measurements to observe the direction and velocity of the current in various levels (up to D'') above the sea-bottom, V_o'', V_s'' and D'' could be determined directly, and then from the equations (11) and (12) one could calculate the two coefficients denoted there by κ and 600. We have therefore a possibility, though perhaps a remote one, of determining these two coefficients, which are fundamental in the theory of ocean-currents.

[25] "On the Influence of the Earth's Rotation on Ocean-Currents," *loc. cit.*, p. 9, equation 7. (V_o' and D' are, in this equation, denoted by V_o and D.)

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